ON POWER SERIES WITH POSITIVE REAL PART IN THE UNIT CIRCLE.*

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1. **Introduction.** Let $\varphi(z)$ be a power series convergent for $|z| < 1$ and such that $\Re\varphi(z) \geqq 0$ in the unit circle. Since the real part of a function holomorphic in the unit circle cannot have a minimum inside the circle without being a constant, it follows that $\Re\varphi(z) > 0$ for $|z| < 1$ unless $\varphi(z)$ is a purely imaginary constant. Disregarding this trivial case, it is seen that multiplying $\varphi(z)$ by a positive constant, we may make $\Re\varphi(0) = \frac{1}{2}$, and subtracting a purely imaginary constant, we may therefore assume $\varphi(z)$ to be of the form

$$\varphi(z) = \tfrac{1}{2} + \sum_{\nu=1}^{\infty} a_\nu z^\nu. \tag{1}$$

The following question now arises: what conditions must the constants $a_1, a_2, \cdots, a_n$ satisfy in order that there shall exist a $\varphi(z)$ of the form (1), convergent for $|z| < 1$, having the given constants as its first n coefficients, and such that $\Re\varphi(z) > 0$ for $|z| < 1$? It has been shown by Carathéodory,† by methods belonging to Minkowski's theory of convex solids, that all $a_1, a_2, \cdots, a_n$ with the required property are interior to or on the boundary of a certain convex solid K_n‡ and may be uniquely represented in parametric form by

$$a_\nu = \lambda_1 e^{-\nu\alpha_1 i} + \lambda_2 e^{-\nu\alpha_2 i} + \cdots + \lambda_n e^{-\nu\alpha_n i} \quad (\nu = 1, 2, \cdots, n) \tag{2}$$

where the α's lie between 0 and 2π (incl.), the λ's are positive or zero, and

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n < 1$$

for points interior to K_n, but

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$$

when the point $a_1, a_2, \cdots, a_n$ is on the boundary of K_n. In the latter

* Read before the American Mathematical Society, September 7, 1921.

† "Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen," Math. Annalen, vol. 64 (1907), pp. 95–115, and "Über den Variabilitätsbereich der Fourier'schen Konstanten von positiven harmonischen Funktionen," Rendiconti del Circolo Matematico di Palermo, vol. 32 (1911), pp. 193–217.

‡ That is, writing $a_\nu = x_\nu + ix_{n+\nu}$ ($\nu = 1, 2, \cdots, n$), the points of rectangular coördinates $x_1, \cdots, x_{2n}$ form a convex solid in Euclidean $2n$-space.

case, $\varphi(z)$ is uniquely determined by the coefficients $a_1, a_2, \cdots, a_n$, and has the form

$$(3) \qquad \varphi(z) = \frac{1}{2}\lambda_1 \frac{e^{\alpha_1 t} + z}{e^{\alpha_1 t} - z} + \frac{1}{2}\lambda_2 \frac{e^{\alpha_2 t} + z}{e^{\alpha_2 t} - z} + \cdots + \frac{1}{2}\lambda_n \frac{e^{\alpha_n t} + z}{e^{\alpha_n t} - z}$$

(where $\lambda_1 \geqq 0, \lambda_2 \geqq 0, \cdots, \lambda_n \geqq 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$).

The convex solid K_n may also be defined by algebraic inequalities involving $a_1, a_2, \cdots, a_n$ and their conjugates $\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n$, as was shown by Toeplitz* and Fischer† through the consideration of certain definite Hermitian forms. Writing $D_0 = 1$ and

$$(4) \qquad D_m = D_m(a_1, a_2, \cdots, a_m) = \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_m \\ \bar{a}_1 & 1 & a_1 & \cdots & a_{m-1} \\ \bar{a}_2 & \bar{a}_1 & 1 & \cdots & a_{m-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \bar{a}_m & \bar{a}_{m-1} & \bar{a}_{m-2} & \cdots & 1 \end{vmatrix}$$

for $m = 1, 2, \cdots, n$, the necessary and sufficient condition that $a_1, a_2, \cdots, a_n$ shall be interior to K_n is

$$(5) \qquad D_0 > 0, \quad D_1 > 0, \quad D_2 > 0, \quad \cdots, \quad D_n > 0,$$

while for a point on the boundary of K_n it is necessary and sufficient that there shall exist a k, where $1 \leqq k \leqq n$, such that

$$(6) \quad D_0 > 0, \quad D_1 > 0, \quad \cdots, \quad D_{k-1} > 0, \quad D_k = D_{k+1} = \cdots = D_n = 0.$$

The preceding results were also obtained independently by F. Riesz.‡

It is the purpose of the present paper to prove all these results by the most elementary function theoretic means, the method of treatment resembling closely that of a preceding paper by the writer§ dealing with a similar problem first solved by Carathéodory and Fejér. The central part of the argument consists in the combination of the process of complete induction with Schwarz' lemma, and thus furnishes a new and not uninteresting example of the fundamental importance of the latter in the theory of functions of a complex variable.

2. **The point set K_n and its correspondence with K_{n-1}.** We begin by recalling some familiar definitions. A sequence of n complex numbers $a_1, a_2, \cdots, a_n$ is called a point (all a's are assumed to be finite). The

* "Über die Fourier'sche Entwickelung positiver Funktionen," Rendiconti del Circolo Matematico di Palermo, vol. 32 (1911), pp. 191–192.

† "Über das Carathéodory'sche Problem, Potenzreihen mit positivem reellen Teil betreffend," ibid., pp. 240–256.

‡ "Sur certains systèmes singuliers d'équations intégrales," Annales de l'Ecole Normale, ser. 3, vol. 28 (1911), pp. 33–62.

§ "On the maximum modulus of an analytic function," these ANNALS, ser. 2, vol. 16 (1914), pp. 77–81.

neighborhood ϵ of a point $a_1, a_2, \cdots, a_n$ consists of all points $a_1', a_2', \cdots, a_n'$ such that

$$|a_1' - a_1| < \epsilon, \qquad |a_2' - a_2| < \epsilon, \qquad \cdots, \qquad |a_n' - a_n| < \epsilon.$$

Consider any point set P. A boundary point of P is any point such that every neighborhood ϵ of this point contains a point belonging to P and also a point not belonging to P; the boundary point itself may or may not belong to P. To every point not on the boundary of P there consequently exists an ϵ such that the neighborhood ϵ of this point consists either of points all belonging to P or of points none of which belongs to P. In the former case, the point is said to be interior to P, and in the latter case, exterior to P. It follows that an interior point belongs to P, while an exterior point does not.

We now define K_n as the set of all points $a_1, a_2, \cdots, a_n$ such that there exists a power series $\varphi(z) = \frac{1}{2} + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$ convergent and with positive real part for $|z| < 1$. Any such $\varphi(z)$ is said to be associated with the point $a_1, a_2, \cdots, a_n$.

Then K_n *contains interior points*, for assuming $|a_1| < 1/2n$, $|a_2| < 1/2n$, $\cdots$, $|a_n| < 1/2n$, the point $a_1, a_2, \cdots, a_n$ belongs to K_n, since the polynomial $\varphi(z) = \frac{1}{2} + a_1 z + a_2 z^2 + \cdots + a_n z^n$ has the required properties on account of $\Re(a_\nu z^\nu) \geqq -|a_\nu z^\nu| > -1/2n$ for $|z| < 1$ and $\nu = 1, 2, \cdots, n$. Consequently, any point $a_1, a_2, \cdots, a_n$ where $|a_1| < 1/4n$, $|a_2| < 1/4n$, $\cdots$, $|a_n| < 1/4n$ has a neighborhood $1/4n$ containing only points of K_n, which proves our statement.

We shall now perform a sequence of transformations which will finally lead to a correspondence between K_n and K_{n-1}. First, consider a $\varphi(z)$ associated with a point of K_n; then $\varphi(z) + \frac{1}{2}$ does not vanish for $|z| < 1$, its real part being greater than $\frac{1}{2}$, and consequently

$$f(z) = \frac{\varphi(z) - \frac{1}{2}}{\varphi(z) + \frac{1}{2}} = a_1 z + \cdots \tag{7}$$

is holomorphic for $|z| < 1$; moreover, the identity

$$\begin{aligned} 1 - |f|^2 &= 1 - f\bar{f} \\ &= 1 - \frac{\varphi - \frac{1}{2}}{\varphi + \frac{1}{2}} \cdot \frac{\bar{\varphi} - \frac{1}{2}}{\bar{\varphi} + \frac{1}{2}} = \frac{\varphi + \bar{\varphi}}{(\varphi + \frac{1}{2})(\bar{\varphi} + \frac{1}{2})} = \frac{2\Re\varphi}{|\varphi + \frac{1}{2}|^2} \end{aligned} \tag{8}$$

shows that $|f(z)| < 1$ for $|z| < 1$ since $\Re\varphi(z) > 0$. Conversely, (7) gives

$$\varphi(z) = \frac{1}{2}\frac{1 + f(z)}{1 - f(z)}, \tag{9}$$

and from (8) and (9) we obtain

$$2\Re\varphi = \frac{1 - |f|^2}{|1 - f|^2}. \tag{10}$$

From (9) and (10) it follows that, for any $f(z)$ holomorphic and less than unity in absolute value for $|z| < 1$, (9) defines a $\varphi(z)$ holomorphic and with positive real part for $|z| < 1$, and if $f(0) = 0$, so that $f(z) = a_1 z + \cdots$, then $\varphi(z) = \frac{1}{2} + a_1 z + \cdots$.

Now let $f(z) = a_1 z + \cdots$ be any function vanishing at the origin, and holomorphic and less than unity in absolute value for $|z| < 1$. Writing

$$g(z) = \frac{1}{z} f(z) = a_1 + \cdots \tag{11}$$

it follows from Schwarz' lemma that

$$|g(z)| \leqq 1 \qquad \text{for} \qquad |z| < 1, \tag{12}$$

and, if $|g(z)| = 1$ for a value of z inside the unit circle, then $g(z)$ is constant $= a_1$, where $|a_1| = 1$. Conversely, any function $g(z)$ holomorphic and less than or equal to unity in absolute value for $|z| < 1$ defines an $f(z) = zg(z)$ holomorphic for $|z| < 1$, and $|f(z)| < 1$ for $|z| < 1$. Thus we always have

$$|a_1| \leqq 1. \tag{13}$$

It now follows that the point set K_1 is defined by (13), so that its boundary points, given by $|a_1| = 1$, belong to K_1, and that with any $a_1 = e^{-\alpha i}$ $(0 \leqq \alpha < 2\pi)$ on the boundary of K_1 there is associated one and only one $\varphi(z)$, namely

$$\varphi(z) = \frac{1}{2} \frac{e^{\alpha i} + z}{e^{\alpha i} - z}. \tag{14}$$

In fact, consider any $\varphi(z) = \frac{1}{2} + a_1 z + \cdots$ holomorphic and of positive real part for $|z| < 1$; then (7) and (11) define a $g(z) = a_1 + \cdots$ satisfying (12), and making $z = 0$ in (12), we obtain (13). Conversely, taking any a_1 such that $|a_1| \leqq 1$, and making $g(z) = a_1$, (12) is satisfied, and (11) and (9) give

$$\varphi(z) = \frac{1}{2} \frac{1 + a_1 z}{1 - a_1 z} \tag{15}$$

as one of the functions satisfying all the conditions imposed on $\varphi(z)$. Moreover, when $|a_1| = 1$, $g(z)$ is uniquely determined and equals a_1, so that (15) is the only φ-function possible, and writing $a_1 = e^{-\alpha i}$, we obtain (14).

Now assume $|a_1| < 1$, then it follows from what precedes that (12) takes the form

$$|g(z)| < 1 \qquad \text{for} \qquad |z| < 1. \tag{12'}$$

Writing

$$(16) \qquad f_1(z) = \frac{g(z) - a_1}{1 - \bar{a}_1 g(z)},$$

we have $|\bar{a}_1 g(z)| < 1$ for $|z| < 1$, so that $f_1(z)$ is holomorphic for $|z| < 1$, and $f_1(0) = 0$; moreover, the identity

$$(17) \qquad 1 - |f_1|^2 = 1 - \frac{g - a_1}{1 - \bar{a}_1 g} \cdot \frac{\bar{g} - \bar{a}_1}{1 - a_1 \bar{g}} = \frac{(1 - |a_1|^2)(1 - |g|^2)}{|1 - \bar{a}_1 g|^2}$$

shows that

$$(18) \qquad |f_1(z)| < 1 \qquad \text{for} \qquad |z| < 1.$$

Conversely, to any $f_1(z)$ holomorphic and less than unity in absolute value for $|z| < 1$, and vanishing at the origin, there corresponds a $g(z)$ obtained from (16)

$$(19) \qquad g(z) = \frac{f_1(z) + a_1}{1 + \bar{a}_1 f_1(z)};$$

this $g(z)$ is holomorphic for $|z| < 1$, $g(0) = a_1$, and $|g(z)| < 1$ for $|z| < 1$, as is seen by interchanging g and f_1 and replacing a_1 by $-a_1$ in (17). Finally, we write

$$(20) \qquad \varphi_1(z) = \frac{1}{2}\frac{1 + f_1(z)}{1 - f_1(z)}, \qquad f_1(z) = \frac{\varphi_1(z) - \frac{1}{2}}{\varphi_1(z) + \frac{1}{2}};$$

it follows from what has been said above in regard to $\varphi(z)$ and $f(z)$ that when $f_1 = b_1 z + \cdots$ is holomorphic and less than unity in absolute value for $|z| < 1$, then $\varphi_1(z) = \frac{1}{2} + b_1 z + \cdots$ is holomorphic and has its real part positive for $|z| < 1$, and vice versa.

We have thus proved that to every $\varphi(z) = \frac{1}{2} + a_1 z + \cdots + a_n z^n + \cdots$, holomorphic and with positive real part for $|z| < 1$, and such that $|a_1| < 1$, there corresponds uniquely, by means of (7), (11), (16) and (20), a $\varphi_1(z) = \frac{1}{2} + b_1 z + \cdots + b_{n-1} z^{n-1} + \cdots$ holomorphic and with positive real part for $|z| < 1$. Conversely, to a given $\varphi_1(z) = \frac{1}{2} + b_1 z + \cdots + b_{n-1} z^{n-1} + \cdots$ holomorphic and with positive real part for $|z| < 1$, and a given a_1 where $|a_1| < 1$, there corresponds uniquely a $\varphi(z) = \frac{1}{2} + a_1 z + \cdots + a_n z^n + \cdots$ holomorphic and of positive real part for $|z| < 1$. It will be necessary for the following to establish the general form of the relation between the coefficients a and b. From (19) and (20) we find

$$\begin{aligned} g(z) &= \frac{(1 + a_1)\varphi_1(z) - \frac{1}{2}(1 - a_1)}{(1 + \bar{a}_1)\varphi_1(z) + \frac{1}{2}(1 - \bar{a}_1)} \\ &= \frac{a_1 + (1 + a_1) b_1 z + \cdots + (1 + a_1) b_{n-1} z^{n-1} + \cdots}{1 + (1 + \bar{a}_1) b_1 z + \cdots + (1 + \bar{a}_1) b_{n-1} z^{n-1} + \cdots} \\ &= a_1 + g_1 z + g_2 z^2 + \cdots + g_{n-1} z^{n-1} + \cdots, \end{aligned}$$

where

$$g_1 = (1 - a_1\bar{a}_1)b_1,$$
$$g_\nu = (1 - a_1\bar{a}_1)b_\nu + G_\nu(a_1, \bar{a}_1, b_1, b_2, \cdots, b_{\nu-1}),$$

for $\nu = 2, 3, \cdots, n - 1$, where G_ν is a polynomial. From (9) and (11) it is seen that

$$\begin{aligned}\varphi(z) &= \frac{1}{2}\frac{1 + zg(z)}{1 - zg(z)}\\ &= \frac{1}{2}\frac{1 + a_1 z + g_1 z^2 + \cdots + g_{n-1}z^n + \cdots}{1 - a_1 z - g_1 z^2 - \cdots - g_{n-1}z^n - \cdots}\\ &= \tfrac{1}{2} + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots,\end{aligned}$$

where $a_2 = g_1 + a_1^2$, $a_\nu = g_{\nu-1} + H_\nu(a_1, g_1, g_2, \cdots, g_{\nu-2})$ for $\nu = 3, 4, \cdots, n$, H_ν being a polynomial. Substituting the expressions of the g's in terms of the b's, we find

$$(21)\qquad \begin{aligned}a_2 &= (1 - a_1\bar{a}_1)b_1 + a_1^2,\\ a_\nu &= (1 - a_1\bar{a}_1)b_{\nu-1} + A_\nu(a_1, \bar{a}_1, b_1, b_2, \cdots, b_{\nu-2})\end{aligned}$$

for $\nu = 3, 4, \cdots, n$, where A_ν is a polynomial. In a similar manner, we obtain from

$$g(z) = \frac{1}{z}\frac{\varphi(z) - \frac{1}{2}}{\varphi(z) + \frac{1}{2}}, \qquad \varphi_1(z) = \frac{1}{2}\frac{1 - a_1 + (1 - \bar{a}_1)g(z)}{1 + a_1 - (1 + \bar{a}_1)g(z)}$$

the formulas

$$(22)\qquad \begin{aligned}b_1 &= \frac{1}{1 - a_1\bar{a}_1}(a_2 - a_1^2),\\ b_\nu &= \frac{1}{(1 - a_1\bar{a}_1)^\nu}B_\nu(a_1, \bar{a}_1, a_2, a_3, \cdots, a_{\nu+1})\end{aligned}$$

for $\nu = 2, 3, \cdots, n - 1$, where B_ν is a polynomial.

We may now summarize the preceding results in the statement that the one-to-one correspondence between the points $a_1, a_2, \cdots, a_n$ for which $|a_1| \neq 1$ and the points $a_1, b_1, b_2, \cdots, b_{n-1}$ defined by (21) and (22) is such that when $a_1, a_2, \cdots, a_n$ belongs to K_n, then $b_1, b_2, \cdots, b_{n-1}$ belongs to K_{n-1} and vice versa. Moreover (21) shows that the a's are bounded when this is the case with the b's (in the exceptional case $|a_1| = 1$ it follows from (14) that $|a_2| = 1, \cdots, |a_n| = 1$) so that K_n is bounded when K_{n-1} is bounded, and since this is evidently the case with K_1 defined by (13), we have the result that

The point set K_n is bounded for every n.

From the continuity of the polynomials contained in (21) and (22) it is seen that if $a_{1\mu}, a_{2\mu}, \cdots, a_{n\mu}$ and $a_{1\mu}, b_{1\mu}, \cdots, b_{n-1,\mu}$ correspond for $\mu = 1, 2, \cdots$ ($a_{1\mu} \neq 1$), and if

$$\lim_{\mu\to\infty} a_{1\mu} = a_1\ (|a_1| \neq 1), \quad \lim a_{2\mu} = a_2, \quad \cdots, \quad \lim a_{n\mu} = a_n,$$
$$\lim b_{1\mu} = b_1, \quad \cdots, \quad \lim b_{n-1,\mu} = b_{n-1},$$

then the points $a_1, a_2, \cdots, a_n$ and $a_1, b_1, b_2, \cdots, b_{n-1}$ also correspond. Assume that a subset of $a_{1\mu}, \cdots, a_{n\mu}$ consists of points belonging to K_n, and another subset of points not belonging to K_n, then the two corresponding subsets of $b_{1\mu}, b_{2\mu}, \cdots, b_{n-1,\mu}$ will and will not belong to K_{n-1} respectively. Hence there corresponds to the boundary point $a_1, a_2, \cdots, a_n$ of K_n (where $|a_1| < 1$) the boundary point $b_1, b_2, \cdots, b_{n-1}$ of K_{n-1} and vice versa. Therefore interior points of K_n and interior points of K_{n-1} also correspond.

We have shown before (see (13)) that all boundary points of K_1 belong to K_1; assuming the same to be true of K_{n-1}, it is also true of K_n. For to a boundary point of K_n for which $|a_1| < 1$ there corresponds a boundary point of K_{n-1} having an associated function $\varphi_1(z)$. From this $\varphi_1(z)$ we form the corresponding $\varphi(z)$ by means of (20), (19), (11) and (9), and this $\varphi(z)$ is associated with the boundary point of K_n from which we started; this boundary point consequently belongs to K_n. Now let $|a_1| = 1$ and make $a_1 = e^{-\alpha i}$, then $\varphi(z)$ is uniquely determined by (14), and it follows that $a_1 = e^{-\alpha i}$, $a_2 = e^{-2\alpha i}, \cdots, a_n = e^{-n\alpha i}$ is the only point belonging to K_n for which $a_1 = e^{-\alpha i}$. This point $a_1, a_2, \cdots, a_n$ is moreover a boundary point, since in its neighborhood there are points where $|a_1| > 1$ and which therefore do not belong to K_n.

Hence K_n is a perfect point set.

3. **Determination of the boundary of K_n and the corresponding functions $\varphi(z)$.** It will now be shown that any point $a_1, a_2, \cdots, a_n$ on the boundary of K_n determines uniquely the associated $\varphi(z)$ which is of the form

$$(23)\qquad \varphi(z) = \tfrac{1}{2}\lambda_1 \frac{e^{\alpha_1 i} + z}{e^{\alpha_1 i} - z} + \tfrac{1}{2}\lambda_2 \frac{e^{\alpha_2 i} + z}{e^{\alpha_2 i} - z} + \cdots + \tfrac{1}{2}\lambda_k \frac{e^{\alpha_k i} + z}{e^{\alpha_k i} - z},$$

where the α's are all different from each other and

$$(24)\quad 0 \leqq \alpha_\nu < 2\pi, \quad \lambda_\nu > 0, \quad \sum \lambda_\nu = 1 \quad (\nu = 1, 2, \cdots, k \text{ and } 1 \leqq k \leqq n)$$

and consequently, expanding both members of (23) in powers of z, that $a_1, a_2, \cdots, a_n$ admit a unique parametric representation

$$(25)\qquad a_\nu = \lambda_1 e^{-\nu\alpha_1 i} + \lambda_2 e^{-\nu\alpha_2 i} + \cdots + \lambda_k e^{-\nu\alpha_k i} \quad (\nu = 1, 2, \cdots, n).$$

Conversely, given any α's and λ's satisfying (24), the point $a_1, a_2, \cdots, a_n$ defined by (25) lies on the boundary of K_n (and the associated function is (23)). Since, by the definition of K_n, the point $a_1, a_2, \cdots, a_m$, where $m < n$, belongs to K_m when $a_1, a_2, \cdots, a_n$ belongs to K_n, the significance of the number k is clearly that $a_1, a_2, \cdots, a_m$ is interior to K_m for $m < k$ but on the boundary of K_m for $k \leqq m \leqq n$.

All these statements have been proved for K_1; now we assume them

to hold for K_{n-1} and prove them for K_n as follows. Let $\varphi(z)$ be associated with the point $a_1, a_2, \cdots, a_n$ on the boundary of K_n where $|a_1| < 1$ (when $|a_1| = 1$ our theorem is already proved by (14)); then the $\varphi_1(z)$ derived from $\varphi(z)$ in the manner explained in the preceding paragraph corresponds to a point $b_1, b_2, \cdots, b_{n-1}$ on the boundary of K_{n-1} and by hypothesis we therefore have

$$\varphi_1(z) = \tfrac{1}{2}\mu_1 \frac{e^{\beta_1 i} + z}{e^{\beta_1 i} - z} + \tfrac{1}{2}\mu_2 \frac{e^{\beta_2 i} + z}{e^{\beta_2 i} - z} + \cdots + \tfrac{1}{2}\mu_{k-1} \frac{e^{\beta_{k-1} i} + z}{e^{\beta_{k-1} i} - z}, \tag{26}$$

where all the β's are different and

$$0 \leqq \beta_\nu < 2\pi, \quad \mu_\nu > 0, \quad \sum \mu_\nu = 1,$$
$$(\nu = 1, 2, \cdots, k-1 \text{ and } 1 \leqq k-1 \leqq n-1),$$

moreover, this $\varphi_1(z)$ is uniquely determined by $b_1, b_2, \cdots, b_{n-1}$, that is, according to (22), by $a_1, a_2, \cdots, a_n$. From (26), (20), (19), (11) and (9) it follows that $\varphi(z)$ is a rational function of degree not exceeding k, and is uniquely determined by $a_1, a_2, \cdots, a_n$.

Let $\overline{\varphi_1(z)}$ be the conjugate of $\varphi_1(z)$, so that

$$\overline{\varphi_1(z)} = \tfrac{1}{2}\mu_1 \frac{e^{-\beta_1 i} + \bar{z}}{e^{-\beta_1 i} - \bar{z}} + \cdots + \tfrac{1}{2}\mu_{k-1} \frac{e^{-\beta_{k-1} i} + \bar{z}}{e^{-\beta_{k-1} i} - \bar{z}};$$

from

$$\frac{e^{-\beta i} + \bar{z}}{e^{-\beta i} - \bar{z}} = \frac{\dfrac{1}{\bar{z}} + e^{\beta i}}{\dfrac{1}{\bar{z}} - e^{\beta i}}$$

it follows that

$$\overline{\varphi_1(z)} = -\varphi_1\left(\frac{1}{\bar{z}}\right) \tag{27}$$

and from (20), (19), (11) and (9) successively

$$\begin{aligned} \overline{f_1(z)} &= \frac{1}{f_1\left(\dfrac{1}{\bar{z}}\right)}, \\ \overline{g(z)} &= \frac{1}{g\left(\dfrac{1}{\bar{z}}\right)}, \\ \overline{f(z)} &= \frac{1}{f\left(\dfrac{1}{\bar{z}}\right)}, \\ \overline{\varphi(z)} &= -\varphi\left(\frac{1}{\bar{z}}\right).* \end{aligned} \tag{28}$$

* The connection of all these equations with Schwarz' principle of reflexion is obvious.

This last equation shows that to a pole z of $\varphi(z)$ inside the unit circle there corresponds a pole $1/\bar{z}$ outside the circle and vice versa, but, by hypothesis, $\varphi(z)$ is holomorphic for $|z| < 1$, and consequently all its poles lie on the unit circle. Let $e^{\alpha i}$ be one of these poles; in its neighborhood we have the expansion

$$\varphi(z) = \tfrac{1}{2}\lambda\left(\frac{e^{\alpha i} + z}{e^{\alpha i} - z}\right)^m + \lambda'\left(\frac{e^{\alpha i} + z}{e^{\alpha i} - z}\right)^{m-1} + \cdots + \lambda^{(m)}\frac{e^{\alpha i} + z}{e^{\alpha i} - z} + P(z - e^{\alpha i})$$

where $\lambda \neq 0$ and P contains positive powers only. Now make

$$z = e^{\alpha i}(1 - \rho e^{\theta i}),$$

then $|z|^2 = 1 - 2\rho\cos\theta + \rho^2$ so that $|z| < 1$ for $-\frac{\pi}{2} + \epsilon \leqq \theta \leqq \frac{\pi}{2} - \epsilon$ and $0 < \rho < 2\sin\epsilon$, where ϵ is as small as we please. Writing $\lambda = |\lambda|e^{\gamma i}$, $0 \leqq \gamma < 2\pi$, the preceding expansion gives

$$\varphi(z) = |\lambda|e^{(\gamma - m\theta)i} \cdot \frac{2^{m-1}}{\rho^m} + \cdots,$$

and since $\Re\varphi(z) > 0$ for $|z| < 1$, it is necessary that $\cos(\gamma - m\theta) \geqq 0$ for $-\frac{\pi}{2} + \epsilon \leqq \theta \leqq \frac{\pi}{2} - \epsilon$, that is, letting ϵ approach zero,

$$\cos(\gamma - m\theta) \geqq 0 \qquad \text{for} \qquad -\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}\cdot$$

When θ varies in this interval of length π, and $m \geqq 2$, then $\gamma - m\theta$ varies over more than π, so that $\cos(\gamma - m\theta) < 0$ for some value of θ in the interval. Hence $m = 1$, and $\gamma - \theta$ varies over an interval of length π, which must coincide with the interval where the cosine is positive, that is, the interval from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, and consequently $\gamma = 0$. Hence λ is positive, the pole $e^{\alpha i}$ is simple, and since $\varphi(z)$ is of degree $\leqq k$, the number k' of α's cannot exceed k, and consequently $\varphi(z) - \sum_1^{k'} \frac{1}{2}\lambda_\nu \frac{e^{\alpha_\nu i} + z}{e^{\alpha_\nu i} - z}$ has no poles and therefore equals a constant c. Now, by hypothesis, $\varphi(0) = \frac{1}{2}$, whence $\varphi(\infty) = -\frac{1}{2}$ by (28); hence $\sum\frac{1}{2}\lambda_\nu + c = \frac{1}{2}$, $-\sum\frac{1}{2}\lambda_\nu + c = -\frac{1}{2}$, so that $c = 0$, $\sum\lambda_\nu = 1$, and

$$\varphi(z) = \sum_1^{k'} \tfrac{1}{2}\lambda_\nu \frac{e^{\alpha_\nu i} + z}{e^{\alpha_\nu i} - z}\cdot$$

It remains to show that $k' = k$. From the preceding expression for $\varphi(z)$, we form $g(z)$; using $\sum\lambda_1 = 1$, we find

$$g(z) = \frac{1}{z}\frac{\varphi(z) - \frac{1}{2}}{\varphi(z) + \frac{1}{2}} = \frac{1}{z}\cdot\frac{\sum_1^{k'}\lambda_\nu\left(\frac{e^{\alpha_\nu i} + z}{e^{\alpha_\nu i} - z} - 1\right)}{\sum_1^{k'}\lambda_\nu\left(\frac{e^{\alpha_\nu i} + z}{e^{\alpha_\nu i} - z} + 1\right)} = \frac{\sum_1^{k'}\frac{\lambda_\nu}{e^{\alpha_\nu i} - z}}{\sum_1^{k'}\frac{\lambda_1 e^{\alpha_\nu i}}{e^{\alpha_\nu i} - z}},$$

so that the degree of $g(z)$ does not exceed $k' - 1$, and by (19) and (20), the degree $k - 1$ of $\varphi_1(z)$ does not exceed $k' - 1$, or $k \leqq k'$. Since it was shown before that $k' \leqq k$, it follows that $k' = k$, and the first part of our theorem is proved.

To prove the second part, viz., that any α's and λ's satisfying (24) and substituted in (25) yield a point $a_1, a_2, \cdots, a_n$ on the boundary of K_n, we note that since the real part of $\dfrac{e^{\alpha i} + z}{e^{\alpha i} - z}$ is positive for $|z| < 1$, the function (23), formed with any α's and λ's satisfying (24), fulfills all the conditions imposed on $\varphi(z)$. Hence the corresponding $a_1, a_2, \cdots, a_n$, given by (25), belongs to K_n, and all that remains to be shown is that this point lies on the boundary of K_n. We observe first that by (24) and (25) $|a_1| < 1$ unless $k = 1$ and $|a_1| = 1$, which case has been dealt with previously. From (23) and (24), (28) follows, and forming $\varphi_1(z)$ by means of (23), (9), (11), (19) and (20), it is seen that (27) is a consequence of (28). From (27), we conclude that to a pole z of $\varphi_1(z)$ inside the unit circle there corresponds a pole $1/\bar{z}$ outside the circle and vice versa, but $\varphi_1(z)$ being holomorphic for $|z| < 1$, all its poles therefore lie on the unit circle. The real part of $\varphi_1(z)$ being positive for $|z| < 1$, we conclude, by the reasoning previously applied to $\varphi(z)$, that all the poles are simple, that their number does not exceed $k - 1$ (since the degree of $g(z)$ does not exceed $k - 1$), and that we have the expansion

$$\varphi_1(z) = \sum_1^{k'-1} \tfrac{1}{2}\mu_\nu \frac{e^{\beta_\nu i} + z}{e^{\beta_\nu i} - z} + c,$$

where $k' \leqq k$, c is a constant and all $\mu_\nu > 0$. From the way $\varphi_1(z)$ is obtained from $\varphi(z)$ defined by (23), it follows that $\varphi_1(0) = \frac{1}{2}$, $\varphi_1(\infty) = -\frac{1}{2}$, so that $c = 0$, $\sum \mu_\nu = 1$. Hence $\varphi_1(z)$ is of the form (26), and since our theorem is assumed to be proved for K_{n-1}, the point $b_1, b_2, \cdots, b_{n-1}$ to which $\varphi_1(z)$ is associated, lies on the boundary of K_{n-1}. From the correspondence between K_n and K_{n-1}, it now follows that $a_1, a_2, \cdots, a_{n-1}$ lies on the boundary of K_n.

4. **Alternative proof of the results of the preceding paragraph.** The proof now to be presented is as simple as the preceding one and has the advantage of showing in addition that the poles $e^{\alpha_1 i}, \cdots, e^{\alpha_k i}$ of $\varphi(z)$ on one hand, and the poles $e^{\beta_1 i}, \cdots, e^{\beta_{k-1} i}$ of $\varphi_1(z)$ together with $e^{\beta_k i} = \dfrac{1 + \bar{a}_1}{1 + a_1}$ on the other, separate each other on the unit circle (except in a limiting case, where $e^{\alpha_k i}$ and $e^{\beta_k i}$ coincide). Eliminating the intermediate functions from (9), (11), (19) and (20), we find

$$(29)\quad \varphi(z) = \frac{1}{2}\,\frac{\dfrac{1}{2}\dfrac{1-\bar{a}_1-(1-a_1)z}{1+\bar{a}_1-(1+a_1)z}+\dfrac{1+\bar{a}_1+(1+a_1)z}{1+\bar{a}_1-(1+a_1)z}\varphi_1(z)}{\dfrac{1}{2}\dfrac{1-\bar{a}_1+(1-a_1)z}{1+\bar{a}_1-(1+a_1)z}+\varphi_1(z)}.$$

To prove the first part of our theorem, assume $\varphi(z)$ to be associated with a point $a_1, a_2, \cdots, a_n$ on the boundary of K_n where $|a_1| < 1$; it follows that $\varphi_1(z)$ has the form (26), and consequently that (27) holds. Writing

$$i\psi(z) = \frac{1}{2}\frac{1-\bar{a}_1+(1-a_1)z}{1+\bar{a}_1-(1+a_1)z}+\varphi_1(z),$$

it is seen from (27) that

$$\overline{\psi(z)} = \psi\left(\frac{1}{\bar{z}}\right),$$

and consequently $\psi(z)$ is real when $|z| = 1$. Making $z = e^{(\beta_\nu+\theta)i}$, where θ is sufficiently small, we now evidently have the following expansions

$$\psi(z) = \mu_\nu\cdot\frac{1}{\theta}+P(\theta)$$

when $e^{\beta_\nu i}$ does not coincide with $e^{\beta_k i} = \dfrac{1+\bar{a}_1}{1+a_1}$;

$$\psi(z) = \frac{1-|a_1|^2}{|1+a_1|^2}\cdot\frac{1}{\theta}+P(\theta)$$

when $\nu = k$ and $e^{\beta_k i}$ does not coincide with any other $e^{\beta_\nu i}$;

$$\psi(z) = \left(\mu_\nu+\frac{1-|a_1|^2}{|1+a_1|^2}\right)\frac{1}{\theta}+P(\theta)$$

when $e^{\beta_\nu i}$ $(\nu < k)$ coincides with $e^{\beta_k i}$. The coefficient of $1/\theta$ being positive in all three cases, it follows that, $e^{\beta_1 i}, \cdots, e^{\beta_{k-1} i}, e^{\beta_k i}$ being arranged in order on the unit circle, the interval between two consecutive ones contains an odd number of zeros of $\psi(z)$. Since the degree of $\psi(z)$ is $k-1$ or k, according as $e^{\beta_k i}$ does or does not coincide with another $e^{\beta_\nu i}$ where $\nu < k$, it follows that each of these intervals, the number of which is $k-1$ or k, contains exactly one simple zero of $\psi(z)$, and that this function has no other zeros. By (29), the poles of $\varphi(z)$ are the k zeros of $\psi(z)$ when $e^{\beta_k i}$ does not coincide with any other $e^{\beta_\nu i}$, but when this is the case, then $e^{\beta_k i}$ is a simple pole of $\varphi(z)$, the other poles being the $k-1$ zeros of $\psi(z)$. Consequently

$$\varphi(z) = \sum_{\nu=1}^{k}\tfrac{1}{2}\lambda_\nu\frac{e^{\alpha_\nu i}+z}{e^{\alpha_\nu i}-z}+c$$

where $e^{\alpha_1 i}, \cdots, e^{\alpha_k i}$ separate and are separated by $e^{\beta_1 i}, \cdots, e^{\beta_{k-1} i}, e^{\beta_k i}$, except when $e^{\beta_k i}$ coincides with another $e^{\beta_\nu i}$, in which case one $e^{\alpha_\nu i}$, say

$e^{\alpha_k i}$, coincides with $e^{\beta_k i}$ and the $k-1$ other $e^{\alpha_\nu i}$ separate and are separated by $e^{\beta_1 i}, \cdots, e^{\beta_{k-1} i}$. The proof that $\sum_1^k \lambda_\nu = 1$ and $c = 0$ is the same as in the preceding paragraph, and we may either use the method given there to show that all λ_ν are positive, or we may use $\overline{\varphi(z)} = -\varphi\left(\frac{1}{\bar{z}}\right)$ to show that all λ's are real, and then make $z = \rho e^{\alpha_\nu i}$ and let ρ approach unity to conclude that $\lambda_\nu > 0$.

To prove the second part of our theorem, we assume $\varphi(z)$ to be of the form (23) with given α's and λ's satisfying (24). Then evidently (28) is true. Compute the corresponding $\varphi_1(z)$; we find from (29)

$$(30)\qquad \varphi_1(z) = \frac{1}{2}\,\frac{\dfrac{1}{2}\dfrac{1-\bar{a}_1-(1-a_1)z}{1+\bar{a}_1-(1+a_1)z} - \dfrac{1-\bar{a}_1+(1-a_1)z}{1+\bar{a}_1-(1+a_1)z}\varphi(z)}{\varphi(z) - \dfrac{1}{2}\dfrac{1+\bar{a}_1+(1+a_1)z}{1+\bar{a}_1-(1+a_1)z}}.$$

Writing

$$i\psi_1(z) = \varphi(z) - \frac{1}{2}\frac{1+\bar{a}_1+(1+a_1)z}{1+\bar{a}_1-(1+a_1)z},$$

it follows from (28) that $\psi_1(z)$ is real for $|z| = 1$. The degree of $\psi_1(z)$ is k or $k+1$ according as $e^{\beta_k i} = \dfrac{1+\bar{a}_1}{1+a_1}$ does or does not coincide with any of the $e^{\alpha_\nu i}$, and it is evident at once that $\psi_1(z) = 0$ for $z = 0$ and $z = \infty$, so that there remain $k-2$ or $k-1$ zeros respectively to be located. Making $z = e^{(\alpha_\nu+\theta)i}$, we have the expansions

$$\psi_1(z) = \lambda_\nu \cdot \frac{1}{\theta} + P(\theta)$$

when $e^{\alpha_\nu i}$ does not coincide with $e^{\beta_k i}$,

$$\psi_1(z) = -(1-\lambda_\nu)\cdot\frac{1}{\theta} + P(\theta)$$

when $e^{\alpha_\nu i}$ coincides with $e^{\beta_k i}$, and for $z = e^{(\beta_k+\theta)i}$

$$\psi_1(z) = -\frac{1}{\theta} + P(\theta)$$

when $e^{\beta_k i}$ does not coincide with any $e^{\alpha_\nu i}$. Hence arranging $e^{\alpha_1 i}, \cdots, e^{\alpha_k i}$ and $e^{\beta_k i}$ in order on the unit circle, obtaining k or $k+1$ intervals according as there is coincidence or not, it follows that the two intervals adjacent to $e^{\beta_k i}$ contain an even number of zeros of $\psi(z)$, the remaining $k-2$ or $k-1$ intervals an odd number. But there were exactly $k-2$ or $k-1$ zeros to be located, and it follows that they are all simple and situated one in

each of the intervals not adjacent to $e^{\beta_k i}$. Now reasoning on the poles of (30) as before on those of (29), we find that $\varphi_1(z)$ has the form

$$\varphi_1(z) = \sum_{\nu=1}^{k-1} \tfrac{1}{2}\mu_\nu \frac{e^{\beta_\nu i} + z}{e^{\beta_\nu i} - z} + c$$

with the separation of the $e^{\alpha_\nu i}$ and the $e^{\beta_\nu i}$ found previously; from (30) it is seen at once that $\varphi_1(0) = \frac{1}{2}$, $\varphi_1(\infty) = -\frac{1}{2}$, hence $\sum_1^{k-1} \mu_\nu = 1$, $c = 0$. We prove as for $\varphi(z)$ that all μ's are positive, and hence $\varphi_1(z)$ has the form (26), being therefore associated with a point $b_1, \cdots, b_{n-1}$ on the boundary of K_{n-1}, and it finally follows that $a_1, a_2, \cdots, a_n$ is on the boundary of K_n.

5. **Proof that K_n is a convex solid, and parametric representation of its interior points.** Let $\varphi_1(z)$ and $\varphi_2(z)$ be two functions associated with the points $a_1', a_2', \cdots, a_n'$ and $a_1'', a_2'', \cdots, a_n''$ both belonging to K_n. The function $\varphi(z) = (1 - t)\varphi_1(z) + t\varphi_2(z)$ where $0 \leqq t \leqq 1$ evidently is holomorphic and of positive real part for $|z| < 1$, and $\varphi(0) = \frac{1}{2}$. Consequently, $\varphi(z)$ is associated with the point $a_1, a_2, \cdots, a_n$, where $a_\nu = (1 - t)a_\nu' + ta_\nu''$, so that this point also belongs to K_n. Therefore K_n is a convex point set, and being perfect, bounded, and containing a $2n$-dimensional neighborhood of the origin as interior points, K_n is a convex $2n$-dimensional solid according to Minkowski's definition.

It is readily seen that when $a_1, a_2, \cdots, a_n$ belongs to K_n, then $ta_1, ta_2, \cdots, ta_n$ is an interior point for $0 \leqq t < 1$. In fact, there exists a neighborhood ϵ of the origin such that all its points belong to K_n; to any point $a_1', a_2', \cdots, a_n'$ such that $|a_\nu' - ta_\nu| < \epsilon(1 - t)$ for $\nu = 1, 2, \cdots, n$ we adjoin another $a_1'', a_2'', \cdots, a_n''$ by the equations $a_\nu' = (1 - t)a_\nu'' + ta_\nu$. It follows that $(1 - t)|a_\nu''| = |a_\nu' - ta_\nu| < \epsilon(1 - t)$ or $|a_\nu''| < \epsilon$, so that $a_1'', a_2'', \cdots, a_n''$ belongs to K_n, and consequently $a_1', a_2', \cdots, a_n'$ also belongs to K_n, since it lies on the segment joining $a_1'', a_2'', \cdots, a_n''$ and $a_1, a_2, \cdots, a_n$. Thus the neighborhood $(1 - t)\epsilon$ of $ta_1, ta_2, \cdots, ta_n$ belongs to K_n, and $ta_1, ta_2, \cdots, ta_n$ is therefore an interior point.

This result may also be expressed as follows: when $a_1, a_2, \cdots, a_n$ is a point interior to K_n but distinct from the origin, then there exists one and only one t, where $0 < t < 1$, such that the point $\frac{a_1}{t}, \frac{a_2}{t}, \cdots, \frac{a_n}{t}$ is on the boundary of K_n. By (25) and (24), this boundary point has the unique parametric representation

$$\frac{a_\nu}{t} = \lambda_1' e^{-\nu\alpha_1 i} + \lambda_2' e^{-\nu\alpha_2 i} + \cdots + \lambda_k' e^{-\nu\alpha_k i} \quad (\nu = 1, 2, \cdots, n)$$

with $0 \leqq \alpha_\nu < 2\pi$, all α's different, $\lambda_\nu' > 0$, $\sum \lambda_\nu' = 1$ $(\nu = 1, 2, \cdots, k)$ and $1 \leqq k \leqq n$. Writing $t\lambda_\nu' = \lambda_\nu$, it is seen that the interior point

$a_1, a_2, \cdots, a_n$ has the unique parametric representation

(25′) $$a_\nu = \lambda_1 e^{-\nu\alpha_1 i} + \lambda_2 e^{-\nu\alpha_2 i} + \cdots + \lambda_k e^{-\nu\alpha_k i} \quad (\nu = 1, 2, \cdots, n)$$

with the α's all different and

(24′) $$a \leqq \alpha_\nu < 2\pi, \quad \lambda_\nu > 0, \quad \sum \lambda_\nu < 1$$
$$(\nu = 1, 2, \cdots, k \text{ and } 1 \leqq k \leqq n).$$

Making all λ's equal to zero, this result holds also for the origin. To prove that conversely the point defined by (25′), from given α's and λ's satisfying (24′), is interior to K_n, we remark that

(23′) $$\varphi(z) = \tfrac{1}{2}\lambda_0 + \tfrac{1}{2}\lambda_1 \frac{e^{\alpha_1 i} + z}{e^{\alpha_1 i} - z} + \tfrac{1}{2}\lambda_2 \frac{e^{\alpha_2 i} + z}{e^{\alpha_2 i} - z} + \cdots + \tfrac{1}{2}\lambda_k \frac{e^{\alpha_k i} + z}{e^{\alpha_k i} - z},$$

where $\lambda_0 = 1 - \lambda_1 - \lambda_2 - \cdots - \lambda_k > 0$, is evidently a φ-function associated with the point $a_1, a_2, \cdots, a_n$ defined by (25′) which therefore belongs to K_n. The point is an interior one, since if it were on the boundary, (23) would give $\varphi(z)$ uniquely and in the form

$$\varphi(z) = \tfrac{1}{2}\lambda_1' \frac{e^{\alpha_1' i} + z}{e^{\alpha_1' i} - z} + \cdots + \tfrac{1}{2}\lambda_m' \frac{e^{\alpha_m' i} + z}{e^{\alpha_m' i} - z},$$

which cannot coincide with (23′) unless $m = k$, $\alpha_\nu' = \alpha_\nu$, $\lambda_\nu' = \lambda_\nu$ and consequently $\lambda_0 = 0$ contrary to (24′). It should be noted that (23′) is not the only $\varphi(z)$ associated with the interior point $a_1, a_2, \cdots, a_n$.

6. **The characterization of K_n by algebraic inequalities involving $a_1, a_2, \cdots, a_n$ and their conjugates:** These inequalities are already stated in (5) and (6), the D's being defined by (4) and $D_0 = 1$. For $n = 1$, these inequalities reduce to $D_1 > 0$ in the interior and $D_1 = 0$ on the boundary of K_1, and since (4) gives $D_1(a_1) = 1 - a_1\bar{a}_1 = 1 - |a_1|^2$, the desired result is obtained immediately by comparison with (13). From what has been said before regarding the correspondence between K_n and K_{n-1}, it is obvious that the inequalities (5) and (6) follow in the general case by complete induction from the identity

(31) $$D_m(a_1, a_2, \cdots, a_m) = (1 - a_1\bar{a}_1)^m D_{m-1}(b_1, b_2, \cdots, b_{m-1}),$$
$$m = 1, 2, \cdots, n,$$

which we shall now proceed to prove. We begin by showing that when $a_1, a_2, \cdots, a_n$ is on the boundary of K_n, then $D_m(a_1, a_2, \cdots, a_m) = 0$ for $m \geqq k$, where k is the integer occurring in the parametric representation (25). In fact, by (25) and (24), the element in the pth column and qth row of the determinant (4) is seen at once to be

$$\sum_{\nu=1}^{k} \lambda_\nu e^{(q-p)\alpha_\nu i},$$

and expanding the determinant in powers of the λ's, we find

$$D_m(a_1, a_2, \cdots, a_m) = \sum_{\nu_1, \nu_2, \cdots, \nu_m+1 = 1, 2, \cdots, k} \lambda_{\nu_1}\lambda_{\nu_2}\cdots\lambda_{\nu_{m+1}} \left| e^{(q-p)\alpha_{\nu_p} i} \right|_{p, q = 1, 2, \cdots, m+1}.$$

Since there are only $k < m + 1$ different α's, any one of the determinants to the right contains the same α_ν in two columns, say the pth and rth, and the latter column is obtained from the former by multiplication by $e^{(p-r)\alpha_\nu p i}$, so that the determinant vanishes, and consequently

$$(32) \qquad D_m(a_1, a_2, \cdots, a_m) = 0 \qquad \text{for} \qquad k \leqq m \leqq n$$

when $a_1, a_2, \cdots, a_n$ is on the boundary of K_n.

Next, we observe that (31) is obtained at once by direct calculation of the determinant (4) for $m = 1$ and $m = 2$, using the expression (21) for a_2 in the latter case.

Now assume the inequalities (5) and (6) proved for $K_1, K_2, \cdots, K_{n-1}$, and that the identity (31) holds for $m = 1, 2, \cdots, n - 1$. Assume $|a_1| < 1$ and that $b_1, b_2, \cdots, b_{n-1}$ satisfies $D_{n-1}(b_1, b_2, \cdots, b_{n-1}) = 0$ and the further conditions

$$(33) \quad D_1(b_1) > 0, \quad D_2(b_1, b_2) > 0, \quad \cdots, \quad D_{n-2}(b_1, b_2, \cdots, b_{n-2}) > 0.$$

Then $b_1, b_2, \cdots, b_{n-1}$ is on the boundary of K_{n-1} (but $b_1, b_2, \cdots, b_{n-2}$ interior to K_{n-2}) and consequently, calculating $a_2, \cdots, a_n$ from (21), the point $a_1, a_2, \cdots, a_n$ is on the boundary of K_n, so that $D_n(a_1, a_2, \cdots, a_n) = 0$ by (32). In other words, taking arbitrary fixed values of $b_1, b_2, \cdots, b_{n-2}$ satisfying (33) and a variable b_{n-1}, and calculating $a_1, a_2, \cdots, a_n$ by (21), then $D_n(a_1, a_2, \cdots, a_n)$ becomes a polynomial in the two variables b_{n-1} and $\bar{b}_{n-1}$, which vanishes whenever the polynomial in the same two variables $D_{n-1}(b_1, b_2, \cdots, b_{n-1})$ vanishes. Consequently the former polynomial is divisible by the latter:

$$(34) \quad \begin{aligned} &D_n(a_1, a_2, \cdots, a_n) \\ &\qquad = \psi(a_1, \bar{a}_1, b_1, \bar{b}_1, \cdots, b_{n-1}, \bar{b}_{n-1}) D_{n-1}(b_1, b_2, \cdots, b_{n-1}) \end{aligned}$$

where ψ is a polynomial in b_{n-1} and $\bar{b}_{n-1}$. By (4), D_n is linear in each of the two variables a_n and $\bar{a}_n$, the coefficient of $a_n\bar{a}_n$ being $- D_{n-2}(a_1, a_2, \cdots, a_{n-2})$, hence using (21), we see that D_n is linear in each of the variables b_{n-1} and $\bar{b}_{n-1}$, the coefficient of $b_{n-1}\bar{b}_{n-1}$ being $- (1 - a_1\bar{a}_1)^2 D_{n-2}(a_1, a_2, \cdots, a_{n-2})$. The coefficient of $b_{n-1}\bar{b}_{n-1}$ in $D_{n-1}(b_1, b_2, \cdots, b_{n-1})$ being $- D_{n-3}(b_1, b_2, \cdots, b_{n-3})$, it follows from (34) that ψ cannot contain b_{n-1} or $\bar{b}_{n-1}$, and comparing coefficients of $b_{n-1}\bar{b}_{n-1}$ on both sides, it is seen that

$$(1 - a_1\bar{a}_1)^2 D_{n-2}(a_1, a_2, \cdots, a_{n-2}) = \psi \cdot D_{n-3}(b_1, b_2, \cdots, b_{n-3}).$$

By hypothesis, (31) is proved for $m = n - 2$, so that

$$D_{n-2}(a_1, a_2, \cdots, a_{n-2}) = (1 - a_1\bar{a}_1)^{n-2} D_{n-3}(b_1, b_2, \cdots, b_{n-3});$$

introducing this in the preceding equation and dividing by D_{n-3} which does not vanish by (33), we find $\psi = (1 - a_1\bar{a}_1)^n$ and (31) is proved for $m = n$ (being an algebraic identity, it evidently also holds when the conditions (33) are not satisfied). The induction proof of the inequalities (5) and (6) is now complete.

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